

МИНИСТЕРСТВО ОБРАЗОВАНИЯ И НАУКИ РФ  
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**Algebra**  
**Part I**  
**Studying-methodical manual**

Recommended  
by Methodical Commission of the Faculty of Mechanics and Mathematics  
for students of Nizhny Novgorod State University  
studying at Master's Program 010200 "Mathematics and Computer Sciences"

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**elementary transformation of the second type** of system (1.1) consists in the follows: i-th and k-th equations interchange places and the other ones remain the same. Obviously, the new system is equivalent to (1.1).

Applying elementary transformations we can reduce system (1.1) to **echelon form (quasi-triangular form)**. Using if necessary elementary transformation of the second type we may assume that  $a_{11} \neq 0$  in system (1.1). Adding 1-th equation multiplied by  $c_i = -a_{i1}/a_{11}$  to i-th equation  $i = 2, \dots, m$  we obtain equivalent system

$$\begin{aligned} a'_{11}x_1 + a'_{12}x_2 + \dots + a'_{1n}x_n &= b'_1, \\ a'_{2p}x_p + \dots + a'_{2n}x_n &= b'_2, \\ \dots & \\ a'_{mp}x_p + \dots + a'_{mn}x_n &= b'_m. \end{aligned} \tag{0.3}$$

Thus, using 1-th equation we eliminate unknown  $x_1$  from all succeeding equations. Some other unknowns might have disappeared as well. Note that the first equation in (1.3) is just the same as in (1.1). After that applying if necessary elementary transformations of the second type we may assume that  $a'_{2p} \neq 0$  in system (1.3) and using the second equation eliminate  $x_p$  from succeeding equations. The first and second equations remain the same. Proceeding the process of elimination of unknowns as long as possible we obtain the following system

$$\begin{aligned} \bar{a}_{11}x_1 + \bar{a}_{12}x_2 + \dots + \bar{a}_{1n}x_n &= \bar{b}_1, \\ \bar{a}_{2p}x_p + \dots + \bar{a}_{2n}x_n &= \bar{b}_2, \\ \bar{a}_{3q}x_q + \dots + \bar{a}_{3n}x_n &= \bar{b}_3, \\ \dots & \\ \bar{a}_{rs}x_s + \dots + \bar{a}_{rn}x_n &= \bar{b}_r, \\ 0 &= \bar{b}_{r+1}, \\ \dots & \\ 0 &= \bar{b}_m. \end{aligned} \tag{0.4}$$

Here  $\bar{a}_{11}, \bar{a}_{2p}, \bar{a}_{3q}, \dots, \bar{a}_{rs} \neq 0, 1 < p < q < \dots < s \leq n$ . We say that system (4) has **echelon form**. It may happen that there are no equations of the form  $0 = \bar{b}$ .

In the process of successive elimination it is convenient to work with rows of extended matrix of system (1.1) instead of equations.

**Theorem 1** (i) *Any linear system may be reduced to echelon form (1.4) by elementary transformations.*

(ii) *A linear system is compatible if and only if its echelon form (1.4) does not contain equations of the type  $0 = \bar{b}$  where  $\bar{b} \neq 0$ .*

Suppose that the echelon form (1.4) of system (1.1) does not contain equations of the type  $0 = \bar{b}$  where  $\bar{b} \neq 0$ , i.e. (1.4)(as well as (1.1))is compatible. The unknowns  $x_1, x_p, x_q, \dots, x_s$  are called **pivotal** or **principal** unknowns. The remain unknowns are called **free** unknowns. Thus, there are  $r$  pivotal unknowns and  $n - r$  free unknowns. We can choose arbitrary values of free variables and substitute them in system (1.4). Since  $\bar{a}_{rs} \neq 0$  we can find the value of  $x_r$  from the r-th equation. Substitute this value for  $x_r$  in the first, the second, ..., the (r-1)-th

equations. Now we find the next pivotal unknown from the  $(r-1)$ -th equation. Proceeding this procedure step by step from the bottom to the top we can find unique values for all principal unknowns. Thus, the values of principal unknowns are uniquely determined by values of free unknowns which may be chosen arbitrary.

The procedure allows to find formulas expressing the principal unknowns through the free unknowns. To avoid the abuse of notations assume that  $x_1, \dots, x_r$  are pivotal unknowns and  $x_{r+1}, \dots, x_n$  are free unknowns. We can find formulas

$$x_1 = f_1(x_{r+1}, \dots, x_n), \dots, x_r = f_r(x_{r+1}, \dots, x_n).$$

Written as  $n$ -tuple they give **the general solution** of system (1.1)

$$\bar{X} = (f_1(x_{r+1}, \dots, x_n), \dots, f_r(x_{r+1}, \dots, x_n), x_{r+1}, \dots, x_n).$$

**Corollary 1** (i) A compatible linear system is determined if and only if  $r = n$ .

(ii) A homogeneous linear system has a non-zero solution if  $m < n$ .

**Example 1.** Find the formula of general solution of linear system

$$\begin{aligned} 6x_1 + 3x_2 + 2x_3 + 3x_4 + 4x_5 &= 5, \\ 4x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= 4, \\ 4x_1 + 2x_2 + 3x_3 + 2x_4 + x_5 &= 0, \\ 2x_1 + x_2 + 7x_3 + 3x_4 + 2x_5 &= 1. \end{aligned} \tag{0.5}$$

We will work with rows of extended matrix of the system

$$\begin{pmatrix} 6 & 3 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 2 & 1 & 0 \\ 2 & 1 & 7 & 3 & 2 & 1 \end{pmatrix}.$$

The first and fourth rows (equations) interchange places

$$\begin{pmatrix} 2 & 1 & 7 & 3 & 2 & 1 \\ 4 & 2 & 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 2 & 1 & 0 \\ 6 & 3 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Add the first row (equation) multiplied by  $(-2)$  to the second and the third rows (equations) and add the 1-th row multiplied by  $(-3)$  to the 4-th

$$\begin{pmatrix} 2 & 1 & 7 & 3 & 2 & 1 \\ 0 & 0 & -13 & -4 & -1 & 2 \\ 0 & 0 & -11 & -4 & -3 & -2 \\ 0 & 0 & -19 & -6 & -2 & 2 \end{pmatrix}.$$

To simplify calculations add the third row multiplied by  $(-1)$  to the second and the fourth rows

$$\begin{pmatrix} 2 & 1 & 7 & 3 & 2 & 1 \\ 0 & 0 & -2 & 0 & 2 & 4 \\ 0 & 0 & -11 & -4 & -3 & -2 \\ 0 & 0 & -8 & -2 & 1 & 4 \end{pmatrix}.$$

Add the second row multiplied by  $(-11/2)$  to the third row and add the second row multiplied by  $(-4)$  to the fourth rows

$$\begin{pmatrix} 2 & 1 & 7 & 3 & 2 & 1 \\ 0 & 0 & -2 & 0 & 2 & 4 \\ 0 & 0 & 0 & -4 & -14 & -24 \\ 0 & 0 & 0 & -2 & -7 & -12 \end{pmatrix}.$$

Add the third row multiplied by  $(-1/2)$  to the fourth row

$$\begin{pmatrix} 2 & 1 & 7 & 3 & 2 & 1 \\ 0 & 0 & -2 & 0 & 2 & 4 \\ 0 & 0 & 0 & -4 & -14 & -24 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The last matrix is the extended matrix of system

$$\begin{aligned} 2x_1 + x_2 + 7x_3 + 3x_4 + 2x_5 &= 1, \\ -2x_3 + 2x_5 &= 4, \\ -4x_4 - 14x_5 &= -24, \\ 0 &= 0. \end{aligned} \tag{0.6}$$

We can see that system (1.6) has an echelon form, it is compatible,  $x_1, x_3, x_4$  are principal unknowns,  $x_2, x_5$  are free unknowns. From the third equation of (1.6)  $x_4 = (-7/2)x_5 + 6$ , from the second equation  $x_3 = x_5 - 2$ . Substituting the formulas for  $x_3$  and  $x_4$  in the first equation obtain  $x_1 = (-1/2)x_2 + 3/4x_5 - 3/2$ . Thus,

$$\bar{X} = (-1/2x_2 + 3/4x_5 - 3/2, x_2, x_5 - 2, -7/2x_5 + 6, x_5)$$

is a general solution of (1.5). To obtain a particular solution of (1.5) choose values for free unknowns, say  $x_2 = 2, x_5 = -1$ , and substitute them to the formula of general solution. We will have solution  $(-13/4, 2, -3, 19/2, -1)$ .

## Chapter 1.2. Symmetric group. Elementary group theory

Let  $M$  be a finite set with  $n$  elements. Since we does not concern the nature of its elements we will identify  $M$  and the set of numbers  $\{1, 2, \dots, n\}$ . A bijective mapping (or a one-to-one correspondence)  $f : M \rightarrow M$  is called a **permutation of degree  $n$** . The set of all permutations of degree  $n$  is denoted by  $S_n$ . The permutation may be written as

$$f = \begin{pmatrix} 1 & 2 & \dots & n \\ i_1 & i_2 & \dots & i_n \end{pmatrix}$$

where  $f(t) = i_t$ . Obviously,  $\{i_1, i_2, \dots, i_n\} = \{1, 2, \dots, n\}$ . There are  $n!$  different permutations of degree  $n$ ,  $|S_n| = n!$ . For mappings  $g : X \rightarrow Y$ ,  $f : Y \rightarrow Z$  a new mapping  $f \circ g : X \rightarrow Z$  called **the composition** of  $f$  and  $g$  is defined,  $f \circ g(t) = f(g(t))$ ,  $t = 1, 2, \dots, n$ . In particular, if  $f, g \in S_n$ , then  $f \circ g \in S_n$  and for

$$g = \begin{pmatrix} 1 & 2 & \dots & n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}$$

$$f \circ g = \begin{pmatrix} 1 & 2 & \dots & n \\ f(j_1) & f(j_2) & \dots & f(j_n) \end{pmatrix}.$$

**Example 2.** Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}.$$

Then

$$f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ f(2) & f(4) & f(3) & f(5) & f(1) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 5 & 4 & 3 \end{pmatrix}.$$

Denote by  $e$  the identity mapping  $e(t) = t$ ,  $t = 1, 2, \dots, n$ . Evidently,  $e \circ f = f \circ e = f$  for any  $f \in S_n$ . For any mappings  $f : A \rightarrow B$ ,  $g : B \rightarrow C$ ,  $h : C \rightarrow D$  the associative law is fulfilled  $(h \circ g) \circ f = h \circ (g \circ f)$ . In particular, the associative law holds for permutations  $f, g, h \in S_n$ . Any permutation  $f$  has an inverse  $f^{-1}$ ,  $f^{-1}(f(i)) = i$ ,  $f(f^{-1}(i)) = i$ , i.e.  $f^{-1} \circ f = f \circ f^{-1} = e$ . For example, for  $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$ ,  $f^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$ . Let  $G$  be a non-empty set. A fixed mapping  $\mu : G \times G \rightarrow G$  is called a **binary operation** on  $G$ .

Let  $G$  be a non-empty set  $G$  with a binary operation  $\mu$ . Denote  $\mu(a, b)$  by  $a * b$ . We will denote binary operation on  $G$  by  $*$  as well. A binary operation is called **associative** if  $(a * b) * c = a * (b * c)$  for any  $a, b, c \in G$ . An element  $e \in G$  is called a **unit element** if  $e * a = a * e = a$  for any  $a \in G$ . A set with a binary operation can have only one unit element.

A set  $G$  with an associative binary operation  $*$  is called a **semigroup** and may be denoted by  $(G, *)$ . A semigroup with a unit is called a **monoid** and may be denoted by  $(G, *, e)$ . An element  $a$  of a monoid  $G$  is called **invertible** if there exists an element  $b \in G$  such that  $a * b = b * a = e$ , here  $e$  is a unit of  $G$ . Such an element  $b$  is called an **inverse** of  $a$ . An element  $a$  of monoid  $G$  can have only one inverse. The inverse of  $a$  is denoted by  $a^{-1}$ .

A monoid  $G$  where any element is invertible is called a **group**. Thus, a **group**  $G$  is a non-empty set with a binary operation  $*$  satisfying the following axioms:



- (1) the operation  $*$  is associative, i.e.  $(a * b) * c = a * (b * c)$  for all  $a, b, c \in G$ ,  
(2)  $G$  has a unit element  $e$ , i.e. such an element that  $a * e = e * a = a$  for every  $a \in G$ ,  
(3) any element  $a \in G$  has an inverse  $a^{-1} \in G$ , i.e. for every  $a \in G$  there exists such an element  $a^{-1} \in G$  that  $a * a^{-1} = a^{-1} * a = e$ . The cardinality of a group (semigroup, monoid)

$G$  is called the **order** of  $G$ . Thus,  $S_n$  is a group of order  $n!$ .

**Remark.** Different notations can be used for binary operation on  $G$ , for example  $\circ, \times, +, \cdot, \square$  etc. If binary operation is denoted by  $+$  then the corresponding semigroup (monoid, group)  $G$  is named **additive**. In this particular case we mean  $0$  (null element, zero) instead of  $e$  (unit) and the opposite element  $-a$  instead of the inverse  $a^{-1}$ . If we use  $\cdot$  for a binary operation then  $(G, \cdot)$  is named **multiplicative** semigroup (monoid, group). In this case the sign  $\cdot$  in the product will be often omitted,  $a \cdot b = ab$ .

Now we can see that  $(S_n, \circ)$  is a group. The identity permutation  $e$  is the unit in  $S_n$ . The inverse mapping  $f^{-1}$  is the inverse of  $f$  in the group  $S_n$ . The group  $(S_n, \circ)$  is called a **symmetric group**. The sign  $\circ$  in the product of permutations will be omitted.

Let  $k_1, k_2, \dots, k_r$  be an ordered set of symbols  $k_i \in M = \{1, \dots, n\}$ . A permutation  $f$  such that  $f(k_1) = k_2, f(k_2) = k_3, \dots, f(k_{r-1}) = k_r, f(k_r) = k_1$  and  $f(j) = j$  when  $j \neq k_i, i = 1, \dots, r$  is called a **cycle** of length  $r$  and is denoted by  $(k_1 k_2 \dots k_r)$ . A cycle of length 2 is called a **transposition**.

Two cycles  $f = (k_1 k_2 \dots k_r)$  and  $g = (p_1 p_2 \dots p_s)$  are called **independent (disjoint)** if  $\{k_1, \dots, k_r\} \cap \{p_1, \dots, p_s\} = \emptyset$ . Two independent cycles commute,  $fg = gf$ .

**Theorem 2** Any permutation  $f \in S_n$  is a product of independent cycles  $f = C_1 C_2 \dots C_m$ . This decomposition is unique up to the order of factors.

**Example 3.** Let  $g = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 6 & 3 & 7 & 4 & 1 & 5 \end{pmatrix}$ . Take any symbol really moved by  $g$ , say 1,  $f(1) = 2, f(2) = 6, f(6) = 1$  and we obtain the cycle (126). The next symbol really moved by  $g$  is 4,  $f(4) = 7, f(7) = 5, f(5) = 4$ . We obtain the second cycle (475). We considered all symbols really moved by  $g$ . Thus,  $g = (126)(475)$ .

The cycle  $(12 \dots m)$  is equal to the following product of transpositions

$$(12 \dots m) = (12)(23) \dots (m-1, m).$$

Note that this product contains  $m - 1$  factors. Now it follows from Theorem 3 that any permutation  $f \in S_n$  has a decomposition into a product of transpositions.

**Theorem 3** Let  $f = t_1 \dots t_k$  be a decomposition of  $f \in S_n$  into a product of transpositions. The number  $\varepsilon_f = (-1)^k$  does not depend on which decomposition is used. Moreover,  $\varepsilon_{fg} = \varepsilon_f \varepsilon_g$ .

The number  $\varepsilon_f$  is called the **parity** ( the **sign**) of permutation  $f$ . A permutation  $f$  is called **even** if  $\varepsilon_f = 1$  and **odd** one if  $\varepsilon_f = -1$ . It follows from the theorem that the product of even permutations is an even permutation and the inverse of even permutation is even as well. Therefore, the set of even permutations with respect to multiplication of permutations as binary operation is a group. The group of even permutations of degree  $n$  is called the **alternating group** and denoted by  $A_n$ . Evidently,  $|A_n| = \frac{1}{2}n!$ .

A non-empty subset  $H$  in a group  $(G, *)$  is called a **subgroup** of  $G$  if  $h_1 * h_2 \in H$  and  $h^{-1} \in H$  for any  $h_1, h_2, h \in H$ . Thus,  $(H, *)$  is a group. We can see that  $A_n$  is a subgroup of  $S_n$ .

Let  $S$  be a subset of a group  $G$ . If any element  $g \in G$  can be written as a product of elements of  $S$  and inverses of elements of  $S$  then we say that  $G$  is **generated** by  $S$ . In this case  $S$  is called the **set of generators** of  $G$ . Thus, the set of transpositions is a set of generators of  $S_n$ .

Let  $G$  and  $H$  be groups. A mapping  $\varphi : G \rightarrow H$  is called a homomorphism of groups if  $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2)$  for any  $g_1, g_2 \in G$ . According to the theorem the mapping of parity  $\varepsilon : S_n \rightarrow \{\pm 1\}$  is a homomorphism of groups.

## Section V. Problems for selfstudy

1. Find the formula of general solution of linear system

$$x_1 + 2x_2 + 3x_3 - 2x_4 + x_5 = 4,$$

$$3x_1 + 6x_2 + 5x_3 - 4x_4 + 3x_5 = 5,$$

$$x_1 + 2x_2 + 7x_3 - 4x_4 + x_5 = 11,$$

$$2x_1 + 4x_2 + 2x_3 - 3x_4 + 3x_5 = 6.$$

2. Find the formula of general solution and a particular solution of linear system

$$6x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_5 = 1,$$

$$3x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 = 3,$$

$$3x_1 + 2x_2 - 2x_3 + x_4 = -7,$$

$$9x_1 + 6x_2 + x_3 + 3x_4 + 2x_5 = 2.$$

3. Find the formula of general solution and a particular solution of linear system

$$6x_1 + 4x_2 + 5x_3 + 2x_4 + 3x_5 = 1,$$

$$3x_1 + 2x_2 + 4x_3 + x_4 + 2x_5 = 3,$$

$$3x_1 + 2x_2 - 2x_3 + x_4 = -7,$$

$$9x_1 + 6x_2 + x_3 + 3x_4 + 2x_5 = 2.$$

4. Solve the linear system

$$10x_1 + 23x_2 + 17x_3 + 44x_4 = 25,$$

$$15x_1 + 35x_2 + 26x_3 + 69x_4 = 40,$$

$$25x_1 + 57x_2 + 42x_3 + 108x_4 = 65,$$

$$30x_1 + 69x_2 + 51x_3 + 133x_4 = 95.$$

5. Find the sign of permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 1 & 4 & 2 & 3 \end{pmatrix}.$$

6. Find the sign of permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 3 & 6 & 5 & 7 & 4 & 2 \end{pmatrix}.$$

7. Find the product of permutations (15)(234).

8. Find the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 1 & 2 \end{pmatrix}.$$

9. Find the product

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}^3.$$

10. Find a permutation  $X$  such that  $AXB = C$  where

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 7 & 4 & 5 & 6 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 1 & 3 & 6 & 4 & 7 & 2 \end{pmatrix}.$$

11. Decompose the permutation into a product of independent cycles

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 8 & 9 & 2 & 1 & 4 & 3 & 6 & 7 \end{pmatrix}.$$

12. Find the values for  $i$  and  $k$  such that the product

$$a_{47}a_{63}a_{1i}a_{55}a_{7k}a_{24}a_{31}$$

enters to a determinant of order 7 with the sign +.

13. Compute the determinant

$$\begin{vmatrix} 2 & -5 & 1 & 2 \\ -3 & 7 & -1 & 4 \\ 5 & -9 & 2 & 7 \\ 4 & -6 & 1 & 2 \end{vmatrix}.$$

14. Compute the determinant

$$\begin{vmatrix} 35 & 59 & 71 & 52 \\ 42 & 70 & 77 & 54 \\ 43 & 68 & 72 & 52 \\ 29 & 49 & 65 & 50 \end{vmatrix}.$$

15. Compute the determinant

$$\begin{vmatrix} 5 & 1 & 2 & 7 \\ 3 & 0 & 0 & 2 \\ 1 & 3 & 4 & 5 \\ 2 & 0 & 0 & 3 \end{vmatrix}.$$

16. Compute the determinant

$$\begin{vmatrix} 1 & n & n & \dots & n \\ n & 2 & n & \dots & n \\ n & n & 3 & \dots & n \\ \cdot & \cdot & \cdot & \dots & \cdot \\ n & n & n & \dots & n \end{vmatrix}.$$

17. Solve the system of equations using Cramer's rule

$$2x_1 + 2x_2 - x_3 + x_4 = 4,$$

$$4x_1 + 3x_2 - x_3 + 2x_4 = 6,$$

$$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12,$$

$$3x_1 + 3x_2 - 2x_3 + 2x_4 = 6.$$

18. Find the product of matrices

$$\begin{pmatrix} 5 & 8 & -4 \\ 6 & 9 & -5 \\ 4 & 7 & -3 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 & 5 \\ 4 & -1 & 3 \\ 9 & 6 & 5 \end{pmatrix}.$$

19. Find the inverse of the matrix

$$\begin{pmatrix} 2 & 5 & 7 \\ 6 & 3 & 4 \\ 5 & -2 & -3 \end{pmatrix}.$$

20. Solve the matrix equation

$$\begin{pmatrix} 2 & -3 & 1 \\ 4 & -5 & 2 \\ 5 & -7 & 3 \end{pmatrix} \cdot X \cdot \begin{pmatrix} 9 & 7 & 6 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

21. Compute expressions

$$a) (2+i)(3+7i) - (1+2i)(5+3i), \quad b) \frac{(5+i)(7-6i)}{3+i}, \quad c) i^{77}, \quad d) i^n, \quad n \in \mathbb{Z}.$$

22. Find the trigonometric form of complex number:

$$a) 7; \quad b) 1+i; \quad c) 1-i; \quad d) 1+i\sqrt{3}; \quad e) \sqrt{3}-i.$$

23. Compute the expressions:

$$a) (1+i)^{1000}, \quad b) \left( \frac{\sqrt{3}+i}{1-i} \right).$$

24. Write as polynomials of  $\sin x$  and  $\cos x$

$$\sin 4x, \quad \cos 4x.$$

25. Compute

$$a) \sqrt[6]{i}; \quad b) \sqrt[3]{1}; \quad c) \sqrt[6]{1}; \quad d) \sqrt[3]{1+i}; \quad e) \sqrt[6]{-27}.$$

27. Let  $\varepsilon_k = \cos \frac{2\pi k}{n} + i \sin \frac{2\pi k}{n}$ ,  $0 \leq k < n$ . Prove that

a)  $\sqrt[n]{1} = \{\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}\}$ ;

b)  $\varepsilon_k = \varepsilon_1^k$ ,  $k = 0, 1, \dots, n-1$ ;

c)  $\sqrt[n]{1}$  is a cyclic group of order  $n$  with respect to the multiplication of complex numbers.

28. Find the rank of the matrix

$$\begin{pmatrix} 8 & 2 & 2 & -1 & 1 \\ 1 & 7 & 4 & -2 & 5 \\ -2 & 4 & 2 & -1 & 3 \end{pmatrix}.$$

29. Find a basis of the system of vectors

$$a_1 = (5, 2, -3, 1), \quad a_2 = (4, 1, -2, 3), \quad a_3 = (1, 1, -1, -2)$$

$$a_4 = (3, 4, -1, 2), \quad a_5 = (7, -6, , -7, 0).$$

30. Find a fundamental system of solutions of the homogeneous linear system

$$\begin{aligned}x_1 + x_2 - 2x_3 + 2x_4 &= 0, \\3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0, \\4x_1 + 5x_2 - 2x_3 + 3x_4 &= 0, \\3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0.\end{aligned}$$

31. Find the sum and the intersection of vector spaces  $U_1 = \langle a_1, a_2, a_3 \rangle, U_2 = \langle b_1, b_2, b_3 \rangle$  where

$$\begin{aligned}a_1 &= (1, 2, 1, -1), a_2 = (2, 3, 1, 0), a_3 = (1, 2, 2, -3), \\b_1 &= (1, 1, 1, 1), b_2 = (1, 0, 1, -1), b_3 = (1, 3, 0, -4).\end{aligned}$$

32. Find a basis of the kernel of the linear mapping given by matrix

$$\begin{pmatrix} 3 & 5 & -4 & 2 \\ 2 & 4 & -6 & 3 \\ 11 & 17 & -8 & 4 \end{pmatrix}.$$

33. Find eigenvalues and eigenvectors of linear transformations given by matrices

$$a) \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}, b) \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}.$$

34. The linear transformation  $\varphi$  has the matrix  $\begin{pmatrix} 3 & 5 \\ 4 & 3 \end{pmatrix}$  in the basis  $a_1 = (1, 2), a_2 = (2, 3)$ .

The linear transformation  $\psi$  has the matrix  $\begin{pmatrix} 4 & 6 \\ 6 & 9 \end{pmatrix}$  in the basis  $b_1 = (3, 1), b_2 = (4, 2)$ .

Find the matrices of  $\varphi + \psi$  and  $\varphi \cdot \psi$  in the basis  $b_1, b_2$ .

35. Find the canonical form of the following quadratic forms

a)  $x_1^2 + x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$ ,

b)  $x_1x_2 + x_1x_3 + x_2x_3$ .

36. Apply the orthogonalization process to the system of vectors

$$a_1 = (1, 2, 2, -1), a_2 = (1, 1, -5, 3), a_3 = (3, 2, 8, -7).$$

37. Find the orthogonal projection of a vector  $x$  on a subspace  $U$ .

$x = (7, -4, -1, 2)$ ,  $U$  is the subspace of solutions of linear system

$$\begin{aligned}2x_1 + x_2 + x_3 + 3x_4 &= 0, \\3x_1 + 2x_2 + 2x_3 + x_4 &= 0, \\x_1 + 2x_2 + 2x_3 - 9x_4 &= 0.\end{aligned}$$

38. Reduce the quadratic form to the principal axes

$$6x_1^2 + 5x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_1x_3.$$

39. Find the orthogonalized basis consisting of eigenvectors of the unitary operator given by

the matrix  $\frac{1}{\sqrt{3}} \begin{pmatrix} 1+i & 1 \\ -1 & 1-i \end{pmatrix}$ .

40. Investigate properties of binary operations on a set  $M$ :
- $M = \mathbb{N}, x * y = x^y$ ;
  - $M = \mathbb{N}, x * y = \gcd(x, y)$ ;
  - $M = \mathbb{Z}, x * y = x - y$ ;
  - $M = \mathbb{R}, x * y = x^2 + y^2$ .
41. Show that  $G = [0, 1)$  is a group with respect to binary operation  $\oplus$  where  $a \oplus b = \{a + b\}$  is the fractional part of  $a + b$ .
42. Let  $(G, \cdot)$  be a group. Show that  $(G, *)$  is a group where  $a * b = b \cdot a$ .
43. Find all subgroups in a) the four-element Klein group b) in  $S_3$ , c) in  $A_4$ .
45. Find the order of an element of a group:
- $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 4 & 3 & 7 & 1 & 2 & 5 & 6 & 10 & 9 & 8 \end{pmatrix} \in S_{10}$ ;
  - $-\frac{1}{2} - \frac{\sqrt{3}}{2}i \in \mathbb{C}^*$  where  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ;
  - $\begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C})$ ;
  - $\begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{2} \end{pmatrix} \in GL_2(\mathbb{F}_3)$ ;
  - $\begin{pmatrix} \bar{1} & \bar{2} \\ \bar{0} & \bar{1} \end{pmatrix} \in GL_2(\mathbb{Z}_6)$ .
46. Find elements of order 2 in the groups:
- $\mathbb{C}^*$ , b)  $S_5$ , c)  $A_5$ .
47. Prove that any group of an even order contains an element of order 2.
48. Find all homomorphisms among the following mappings  $f : \mathbb{C}^* \rightarrow \mathbb{R}^*$  :
- $f(z) = |z|$ , b)  $f(z) = 2|z|$ , c)  $f(z) = \frac{1}{|z|}$ , d)  $f(z) = 1$ , e)  $f(z) = |z|^2$ , f)  $f(z) = 1 + |z|$ , g)  $f(z) = 2$ .
49. find all homomorphisms  $\mathbb{Z}_6 \rightarrow \mathbb{Z}_6$ .
50. Find the group of automorphisms of a group:
- $\mathbb{Z}$ , b)  $\mathbb{Z}_p$ ,  $p$  is a prime number, c)  $S_3$ .
51. Prove that group  $S_3$  acts by conjugations on the subset  $M = \{(12)(34), (13)(24), (14)(23)\}$ . Thus, the homomorphism  $\Phi : S_4 \rightarrow S_3$  is defined. Find  $\text{Ker}\Phi$  and  $\Im\Phi$ .
52. Find the quotient group:
- $\mathbb{R}^*/\mathbb{R}^+$ ;
  - $\mathbb{C}^*/\mathbb{R}^+$ ;
  - $\mathbb{C}^*/\mathbb{T}^1$  where  $\mathbb{T}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ ;
  - $\mathbb{T}^1/\mathbb{U}_n$  where  $\mathbb{U}_n = \{z \in \mathbb{C} \mid z^n = 1\}$ ;
  - $GL_n(\mathbb{R})/SL_n(\mathbb{R})$ ;
  - $4\mathbb{Z}/12\mathbb{Z}$ .
53. Using Sylow theorems prove that
- any group of order 15 is a cyclic group;
  - any group of order 36 is not a simple group.
54. Prove that any  $p$ -group is solvable.
55. Decompose the groups into the direct sum:
- $\mathbb{Z}_6$ , b)  $\mathbb{Z}_{12}$ , c)  $\mathbb{Z}_{60}$ .
56. Prove that the group  $D_n = \langle a, b \mid a^2, b^2, (ab)^n \rangle$  is a group of order  $2n$ .
57. Prove that
- $$G = \langle a, b \mid a^2, b^2 \rangle \cong H = \left\{ \begin{pmatrix} \pm 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}.$$
58. Find all up to isomorphisms abelian group of order 27.

59. Find all up to isomorphism abelian groups of order 36.
60. Find out if the following groups are isomorphic:
- $\mathbb{Z}_6 \oplus \mathbb{Z}_{36}$  and  $\mathbb{Z}_{12} \oplus \mathbb{Z}_{18}$ ;
  - $\mathbb{Z}_6 \oplus \mathbb{Z}_{36}$  and  $\mathbb{Z}_9 \oplus \mathbb{Z}_{24}$ .
61. Which of the following sets of numbers form a ring with respect to the usual operations of addition and multiplication:
- $\mathbb{Z}$ , 2)  $n\mathbb{Z}$ , 3) the set of non-negative integers, 4)  $\mathbb{Q}$ , 5)  $\{x + y\sqrt{2} | x, y \in \mathbb{Q}\}$ , 6)  $\{x + y\sqrt[3]{2} | x, y \in \mathbb{Q}\}$ , 7)  $\{x + y\sqrt[3]{2} + z\sqrt[3]{4} | x, y, z \in \mathbb{Q}\}$ , 8)  $\mathbb{Z}[i] = \{x + iy | x, y \in \mathbb{Z}\}$ ?

62. Which of the following sets of functions form a ring with respect to the usual operations of addition and multiplication:

    - the set  $C[a, b]$  of all continuous real functions on the closed interval  $[a, b]$ ;
    - the set of all real functions equal to zero on a fixed subset  $A \subset \mathbb{R}$ ;
    - the set of all trigonometric polynomials

$$\left\{ a_0 + \sum_{k=1}^n (a_k \cos kx + b_k \sin kx) \mid n \in \mathbb{N}, a_k, b_k \in \mathbb{R} \right\}.$$

63. Find all ideals of rings  $\mathbb{Z}$ ,  $K[x]$  where  $K$  is a field.
64. Show that the rings  $\mathbb{Z}[x]$  and  $K[x, y]$  ( $K$  is a field) are not principal ideals rings.
65. Prove that
- $F[x]/(x - a) \cong F$  where  $F$  is a field;
  - $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$ ;
  - $\mathbb{R}[x]/(x^2 + x + 1) \cong \mathbb{C}$ .
66. Let  $K$  be a field. Show that the linear mapping  $\varphi : M_s(K) \otimes_K M_t(K) \longrightarrow M_{st}(K)$ , such that  $\varphi(E_{ij} \otimes E_{rk}) = E_{i+s(r-1), j+s(k-1)}$ ,  $1 \leq i, j \leq s$ ,  $1 \leq r, k \leq t$  is an isomorphism of algebras.
67. Prove that the fields  $\mathbb{Q}$ ,  $\mathbb{R}$  have no automorphisms different from the identity mapping.
68. For what  $n = 2, 3, 4, 5, 6, 7, 8, 9$  there exists a field consisting of  $n$  elements?
69. Solve the equations in  $\mathbb{Z}_{11}$ ; 1)  $x^2 = 5$ , 2)  $x^7 = 7$ , 3)  $x^3 = a$ , 4)  $x^2 + 3x + 7 = 0$ .
70. Find the minimal polynomials for elements
- $\sqrt{2}$  over  $\mathbb{Q}$ , 2)  $\sqrt{2} + \sqrt{3}$  over  $\mathbb{Q}$ , 3)  $1 + \sqrt{2}$  over  $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ .
71. Find the Galois group of fields  $\mathbb{Q}(\sqrt{2})$ ,  $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ ,  $\mathbb{Q}(\sqrt[3]{2})$  over  $\mathbb{Q}$ .
72. Find all commutative ideals of the group algebra  $\mathbb{C}[G]$  for 1)  $G = S_3$ , 2)  $G = D_5$ .
73. Find a basis of the center of the group algebras of the groups  $S_3$ ,  $A_4$ .
74. Find the character of the representation  $\rho$  of  $S_n$  on  $\mathbb{R}^n$  such that  $\rho(\pi_0(e_i)) = e_{\pi(i)}$  where  $\{e_1, \dots, e_n\}$  is the standard basis of  $\mathbb{R}^n$ .
75. Create the table of characters of the group  $S_3$ .
76. Let  $A$  be an algebra over a field  $K$ . A linear mapping  $D : A \longrightarrow A$  such that  $D(ab) = D(a)b + aD(b)$  is called a derivation of algebra  $A$ . Denote by  $Der A$  the linear space of all derivations of an algebra  $A$ . Show that  $Der A$  is a Lie algebra with respect to multiplication  $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$ .
77. Let  $A = A_{\bar{0}} \oplus A_{\bar{1}}$  be a two graded associative algebra over a field  $K$ , i.e.  $A_i A_j \subset A_{i+j}$  (the sum modulo 2). Let  $[a, b] = ab - (-1)^{ij}ba$  where  $a \in A_i$ ,  $b \in A_j$ .
- Prove that for any homogeneous elements  $a_i \in A_i$ ,  $b \in A_j$ ,  $c \in A_k$  we have
- $[a, b] = -(-1)^{ij}[a, b]$  (graded skew-symmetric),
  - $(-1)^{ki}[a, [b, c]] + (-1)^{ij}[b, [c, a]] + (-1)^{jk}[c, [a, b]] = 0$  (graded Jacobi identity).
- A 2-graded algebra with a multiplication satisfying conditions 1) - 2) is called a Lie superalgebra.



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